

Families of Small Regular Graphs of Girth 5

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Abstract

In this paper we obtain $(q + 3)$ -regular graphs of girth 5 with fewer vertices than previously known ones for $q = 13, 17, 19$ and for any prime $q \geq 23$ performing operations of reductions and amalgams on the Levi graph B_q of an elliptic semiplane of type \mathcal{C} . We also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} using the same technique.

1 Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [12].

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the length $g = g(G)$ of its shortest circuit. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v . A graph is called *k-regular* if all its vertices have the same degree k , and *bi-regular* or (k_1, k_2) -*regular* if all its vertices have either degree k_1 or k_2 . A (k, g) -*graph* is a k -regular graph of girth g and a (k, g) -*cage* is a (k, g) -graph with the smallest possible number of vertices. The necessary condition obtained from the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices of a (k, g) -graph, known as the Moore bound.

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$$n_0(k, g) = \begin{cases} 1 + k + k(k-1) + \dots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \dots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Biggs [10] calls *excess* of a (k, g) -graph G the difference $|V(G)| - n_0(k, g)$. Cages have been intensely studied since they were introduced by Tutte [30] in 1947. Erdős and Sachs [16] proved the existence of a (k, g) -graph for any value of k and g . Since then, most of the work carried out has been focused on constructing smallest (k, g) -graphs (see e.g. [1, 2, 4, 5, 6, 7, 9, 13, 17, 19, 22, 25, 27, 28, 32]). Biggs is the author of an impressive report on distinct methods for constructing cubic cages [11]. Royle [29] keeps a web-site in which all the cages known so far appear. More details about constructions on cages can be found in the surveys by Wong [32], by Holton and Sheehan [23, Chapter 6], or the recent one by Exoo and Jajcay [18].

A *partial plane* is an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L}, |)$ in which two distinct points are incident with at most one line. In an incidence structure a *flag* is an incident point line pair $p_1|l_1$, an *anti-flag* is a non-incident point line pair $p_1 \nmid l_1$, two lines are *parallel* if there is no point incident with both, and, dually, two points are *parallel* if there is no line incident with both.

A v_k -configuration or a configuration of type v_k is a partial plane consisting of v points and v lines such that each point and each line are incident with k lines and k points, respectively. A finite *elliptic semiplane* of order $k-1$ is a v_k -configuration satisfying the following axiom of parallels: for each anti-flag $p_1 \nmid l_1$, there exists at most one line l_2 incident with p_1 and parallel to l_1 , and at most one point p_2 incident with l_1 and parallel to p_1 [15, 21].

A *Baer subset* of a finite projective plane P is either a Baer subplane B or, for a distinguished point-line pair (p, l) , the union $B(p, l)$ of all lines and points incident with p and l , respectively. We write $B(p|l)$ or $B(p \nmid l)$, according to the incidence or non-incidence of p and l . It was already known to Dembowski [15] that elliptic semiplanes are obtained by deleting a Baer subset from a projective plane. We call any such elliptic semiplane *Desarguesian* if the projective plane from which it is constructed is so. In [15] Dembowski classified elliptic semiplanes into five types. In this paper we will only be concerned with those of type \mathcal{C} , which are $\mathcal{C}_q = PG(2, q) - B(p|l)$, i.e. the complement of a Baer subset $B(p|l)$ in a desarguesian projective plane $PG(2, q)$, for each prime power q . Hence the elliptic semiplane of type \mathcal{C}_q is also a configuration of type $(q^2)_q$.

The *Levi graph* or *incidence graph* G of an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L}, |)$, is a bipartite graph with $V(G) = V_1 \cup V_2$, where $V_1 = \mathcal{P}$ and $V_2 = \mathcal{L}$ and two vertices are adjacent in G if and only if the corresponding point and line are incident in \mathcal{I} . Recall that the Levi graph of a finite projective plane is a $(k, 6)$ -cage, attaining Moore's bound, i.e. these are Moore graphs [32].

In this paper we obtain $(q+3)$ -regular graphs of girth 5 with fewer vertices than previously known ones (cf. [24, 20]) for $q = 13, 17, 19$ and for any prime $q \geq 23$ performing

operations of reductions (cf. Section 3) on the Levi graph B_q of \mathcal{C}_q and then amalgams with bi-regular graphs (cf. Section 4) into the obtained reduced graph or B_q itself. We also obtain a new 13-regular graph of girth 5 on 236 vertices from B_{11} using the same technique.

2 Preliminaries

Throughout the paper we will use the following notation when dealing with the elliptic semiplane of type \mathcal{C}_q .

In $PG(2, q)$, choose p and l to be the point and line at infinity, respectively. Then, in \mathcal{C}_q it is possible to choose the affine coordinates (x, y) , for the points, and $[m, b]$ for the lines $\{x, y, m, b\} \in GF(q)$, which imply that the incidence between a point and a line is given by the equation $y = mx + b$. Recall that in \mathcal{C}_q vertical lines have been deleted from $PG(2, q)$ along with the point at infinity, the line at infinity and all its points.

Define the sets $P_i = \{(i, y) \mid y \in GF(q)\}$ for $i \in GF(q)$ and $L_j = \{[j, b] \mid b \in GF(q)\}$ for $j \in GF(q)$. These sets correspond to the partition of the points and lines of \mathcal{C}_q into parallel classes, according to the axiom of parallels for elliptic semiplanes. Note also that if $(x, y) \parallel [m, b]$ then $(x, y + a) \parallel [m, b + a]$ for any $a \in GF(q)$.

The following properties of the Levi graph B_q of \mathcal{C}_q are well known and they will be fundamental throughout the paper.

Proposition 2.1 *Let B_q be the Levi graph of \mathcal{C}_q then:*

- (i) *It is q -regular, bipartite, vertex transitive, of order $2q^2$ and has girth 6;*
- (ii) *It admits a partition $V_1 = \bigcup_{i=0}^{q-1} P_i$ and $V_2 = \bigcup_{j=0}^{q-1} L_j$ of its vertex set;*
- (iii) *Each block P_i is connected to each block L_j by a perfect matching, for $i, j \in GF(q)$;*
- (iv) *Each vertex in P_0 and L_0 is connected straight to all its neighbours in B_q , meaning that for $p = (0, y)$, $N(p) = \{[i, y] \mid i \in GF(q)\}$ and analogously for $l = [0, b]$, $N(l) = \{(j, b) \mid j \in GF(q)\}$;*
- (v) *The other matchings between P_i and L_i are twisted and the rule can be defined algebraically in $GF(q)$.*

For further information regarding these properties and for constructions of the adjacency matrix of B_q as a block $(0, 1)$ -matrix please refer to [3, 8].

3 Reductions

In this section we will describe two reduction operations that we perform on the graph B_q .

REDUCTION 1 Remove vertices from P_0 and L_0 .

Let $T \subseteq S \subseteq GF(q)$, $S_0 = \{(0, y) | y \in S\} \subseteq P_0$, $T_0 = \{[0, b] | b \in T\} \subseteq L_0$ and $B_q(S, T) = B_q - S_0 - T_0$.

Lemma 3.1 *Let $T \subseteq S \subseteq GF(q)$. Then $B_q(S, T)$ is bi-regular with degrees $(q-1, q)$ of order $2q^2 - |S| - |T|$. Moreover, the vertices $(i, t) \in V_1$ and $[j, s] \in V_2$, for each $i, j \in GF(q) - \{0\}$, $s \in S$ and $t \in T$ are the only vertices of degree $q-1$ in $B_q(S, T)$, together with $[0, s] \in V_2$ for $s \in S - T$ if $T \subsetneq S$.*

Proof It is an immediate consequence of Remark 2.1 (i), (v). ■

REDUCTION 2 Remove pairs of blocks (P_i, L_i) from B_q .

Let $u \in \{1, \dots, q-1\}$. Define $B_q(u) = B_q - \bigcup_{i=1}^u (P_{q-i} \cup L_{q-i})$ the graph obtained from B_q by deleting the last u pairs of blocks of vertices P_i, L_i . and $B_q(S, T, u) = B_q - S_0 - T_0 - \bigcup_{i=1}^u (P_{q-i} \cup L_{q-i})$.

Lemma 3.2 *Let $u \in \{0, \dots, q-1\}$. Then, the graph $B_q(u)$ is $(q-u)$ -regular of order $2(q^2 - qu)$ and the graph $B_q(S, T, u)$ is bi-regular with degrees $(q-u-1, q-u)$ and order $2(q^2 - qu) - |S| - |T|$. Moreover, the vertices $(i, t) \in V_1$ and $[j, s] \in V_2$, for each $i, j \in GF(q)$, $s \in S$ and $t \in T$ are the only vertices of degree $q-u-1$ in $B_q(S, T, u)$, together with $[0, s] \in V_2$ for $s \in S - T$ if $T \subsetneq S$.*

Proof It is immediate from Remark 2.1 (i), (iv) and Lemma 3.1. ■

Note that, $B_q(u) = B_q$ and $B_q(S, T, u) = B_q(S, T)$ when $u = 0$.

4 Amalgams

In this section we will describe amalgam operations that can be performed on the reduced graph $B_q(S, T, u)$ or on B_q itself.

Let Γ_1 and Γ_2 be two graphs of the same order and with the same label on their vertices. In general, an *amalgam of Γ_1 into Γ_2* is a graph obtained adding all the edges of Γ_1 to Γ_2 .

Let P_i and L_i be defined as in Section 2. Consider the graph $B_q(S, T, u)$, for some $T \subseteq S \subseteq GF(q)$ and $u \in \{0, \dots, q-1\}$. Let $S_0 \subseteq P_0$, $T_0 \subseteq L_0$ as in Reduction 1, and let $P'_0 := P_0 - S_0$ and $L'_0 := L_0 - T_0$ be the blocks in $B_q(S, T, u)$ of order $q - |S|$ and $q - |T|$, respectively.

Let H_1, H_2, G_i , for $i = 1, 2$, be graphs of girth at least 5 and order $q - |S|$, $q - |T|$ and q , respectively. Let H_1 be a k -regular graph. If $|S| = |T|$, let H_2 be k -regular and otherwise let it be $(k, k+1)$ -regular, with $|S - T|$ vertices of degree $k+1$. If $T = \emptyset$, let G_1 be a

k -regular graph and otherwise let it be $(k, k+1)$ -regular with $|T|$ vertices of degree $k+1$. Finally, let G_2 be a $(k, k+1)$ -regular with $|S|$ vertices of degree $k+1$.

We define $B_q^*(S, T, u)$ to be the *amalgam* of H_1 into P'_0 , H_2 into L'_0 , G_1 into P_i and G_2 into L_i , for $i \in \{1, \dots, q-u-1\}$ and $u \in \{0, \dots, q-2\}$. We also define $B_q^*(S, T, q-1)$ to be the amalgam of H_1 into P'_0 , H_2 into L'_0 .

To simplify notation in our results, we label P_i and L_i as in Section 2, but assume that the labellings of H_1, H_2, G_1 and G_2 , correspond to the second coordinates of P'_0, L'_0, P_i and L_i respectively for $i \in \{1, \dots, q-u-1\}$ and $u \in \{0, \dots, q-2\}$. Suppose also that the vertices of degree $k+1$, if any, in H_2, G_1 and G_2 are labelled in correspondence with the second coordinates of $S-T, T$ and S , respectively.

With such a labelling, let ab be an edge in H_1, H_2, G_1 or G_2 , and define the *weight* or the *Cayley Color* of ab to be $\pm(b-a) \in \mathbb{Z}_q^*$. Let \mathcal{P}_ω be the set of weights in H_1 and G_1 , and let \mathcal{L}_ω be the set of weights in H_2 and G_2 .

The following result is a special case of [20, Theorem 2.8] for the coordinates we have chosen for \mathcal{C}_q (cf. Section 2). On the other hand, it generalizes such a Theorem since we delete vertices from P_0 and L_0 , pairs of blocks P_i, L_i and amalgam with graphs which are not regular, but chosen in such a way that the obtained amalgam is regular.

Theorem 4.1 *Let $T \subseteq S \subseteq GF(q)$, $u \in \{0, \dots, q-1\}$. Let H_1, H_2, G_1 and G_2 be defined as above and suppose that the weights $\mathcal{P}_\omega \cap \mathcal{L}_\omega = \emptyset$. Then the amalgam $B_q^*(S, T, u)$ is a $(q+k-u)$ -regular graph of girth at least 5 and order $2(q-u) - |S| - |T|$.*

Proof The order and the regularity of $B_q^*(S, T, u)$ follow from Lemma 3.2 and the choice of H_1, H_2, G_1 and G_2 . Note that the vertices of L_i , with degree $q-u-1$ in $B_q(S, T, u)$, have degree $k+1$ in G_2 , which add up to to degree $q+k-u$ in $B_q^*(S, T, u)$, for $i \in \{1, \dots, q-u-1\}$. Similarly for the vertices in L_0 and for those in P_i , for $i \in \{1, \dots, q-u-1\}$.

Let C be the shortest circuit in $B_q^*(S, T, u)$ and suppose, by contradiction, that $|C| \leq 4$. Therefore, $C = (xyz)$ or $C = (wxyz)$. Since B_q has girth 6 and H_1, H_2, G_1, G_2 have girth at least 5, then C cannot be completely contained in B_q or in some H_i or G_i for $i = 1, 2$. Then, w.l.o.g. the path xyz in C is such that $x, y \in P_i$ and $z \in L_m$ for some $i, m \in GF(q)$. Since the edges between P_i and L_m form a matching, then $xz \notin E(B_q)$ and hence $xz \notin E(B_q^*(S, T, u))$. Thus $|C| > 3$ and we can assume $|C| = 4$ and $C = (wxyz)$.

If $w \in P_i$, by the same argument, $wz \notin E(B_q^*(S, T, u))$ and we have a contradiction. There are no edges between P_i and P_j in $B_q^*(S, T, u)$, so $w \notin P_j$ for $j \in GF(q) - \{i\}$, which implies that $w \in L_n$ for some $n \in GF(q)$. If $n \neq m$, we have a contradiction since there are no edges between L_m and L_n in $B_q^*(S, T, u)$. Therefore $x, y \in P_i$ and $w, z \in L_m$. Let $x = (i, a), y = (i, b), z = [m, c]$ and $w = [m, d]$ as in the labelling chosen in Section 2. Then $wx, yz \in E(B_q^*(S, T, u))$ imply that $a = m \cdot i + d$ and $b = m \cdot i + c$, respectively, which give

$b - a = c - d$. On the other hand $xy, wz \in E(B_q^*(S, T, u))$ implies that $ab \in E(H_1) \cup E(G_1)$ and $cd \in E(H_2) \cup E(G_2)$, so $\pm(a - b) \in \mathcal{P}_\omega$ and $\pm(c - d) \in \mathcal{L}_\omega$, a contradiction, since by hypothesis $\mathcal{P}_\omega \cap \mathcal{L}_\omega = \emptyset$. ■

Remark 4.2 *In most cases the graph $B_q^*(S, T, u)$ has girth exactly 5. We describe two cases that we will use in Sections 5 and 6.*

(i) *If some H_i or G_i contains a 5-circuit, for $i \in \{1, 2\}$, then so does $B_q^*(S, T, u)$.*

(ii) *Let $t \in GF(q)$ be the smallest weight of an edge in some H_i , say w.l.o.g. in H_1 . If $t = 1$ and $u < q - 1$ then $((0, i), (0, j), [1, j], (1, i), [0, i])$ is a 5-circuit in $B_q^*(S, T, u)$. If $t > 1$ then $((0, i), (0, j), [1, j], (t, i), [0, i])$ is a 5-circuit in $B_q^*(S, T, u)$ as long as the t^{th} -pair of blocks from $B_q^*(S, T)$ is not deleted, i.e. $u < q - t$.*

All the graphs constructed in the next sections have girth exactly 5 since either some H_i or G_i contains a 5-circuit, for $i \in \{1, 2\}$, or $1 \in P_\omega$.

5 New Regular Graphs of Girth 5

In this section we will construct new $(q + 3)$ -regular graphs of girth 5, for any prime $q \geq 23$, applying reductions and amalgams to the graph B_q . In each case we will specify the sets S and T of vertices to be deleted from P_0 and L_0 and the graphs H_1, H_2, G_1, G_2 to be used for the amalgam into $B_q^*(S, T, u)$. For $u = 0$, all the graphs $B_q^*(S, T, u)$ constructed in this section have two vertices less than the ones that appear in [24, 20].

Recall that every prime q is either congruent to 1 or 5 modulo 6. We will now treat these two cases separately, when $q = 6n + 1$ or $q = 6n + 5$ is a prime.

5.1 Construction for primes $q = 6n + 1$

Throughout this subsection we will consider $n \geq 5$. The smaller cases will be treated in Section 6. Let H_1 and H_2 be two graphs of order $q - 1$ with the vertices labeled from 1 through $6n$, and partitioned into $W_1 = \{1, 2, \dots, 3n\}$ and $W_2 = \{3n + 1, \dots, 6n\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows:

Set	Edges	Description
A_1	$\{(i, i + 1) i = 1, \dots, 3n - 1\} \cup \{(3n, 1)\}$	(3n)-circuit with weights 1 and $3n - 1$
B_1	$\{(i, i + 2) i = 3n + 1, \dots, 6n - 2\} \cup \{(6n - 1, 3n + 1), (6n, 3n + 2)\}$	one or two circuits according to the parity of n , with weights 2 and $3n - 2$
C_1	$\{(i, 3n + i) i = 1, \dots, 3n\}$	Prismatic edges between W_1 and W_2 of weight $3n$

The graph H_1 is cubic and has weights $\pm\{1, 2, 3n - 2, 3n - 1, 3n\}$.

Lemma 5.1 *The graph H_1 has girth 5.*

Proof Let C be the shortest circuit in H_1 . If C is a subgraph of either $H_1[W_1]$ or $H_1[W_2]$ then $|C| \geq 5$, since $H_1[W_1]$ has girth at least 15 and $H_1[W_2]$ has girth at least 9. Otherwise, there is a path xyz in C such that either $x, y \in W_1$ and $z \in W_2$ or $x \in W_1$ and $y, z \in W_2$. The first case has the following subcases:

- (i) $x = 1, y = 3n, z = 6n$
- (ii) $x = i, y = i - 1, z = 3n + i - 1$, for $i = 2, \dots, 3n$
- (iii) $x = i, y = i + 1, z = 3n + i + 1$, for $i = 1, \dots, 3n - 1$
- (iv) $x = 3n, y = 1, z = 3n + 1$

The second case has similar subcases. If we show that $z \notin N_{H_1}(x)$ then $|C| \neq 3$, and if $y = N_{H_1}(x) \cap N_{H_1}(z)$ then $|C| \neq 4$. In subcase (i) the neighbourhoods of x and z in H_1 are $N_{H_1}(x) = \{2, 3n, 3n + 1\}$ and $N_{H_1}(z) = \{3n, 3n + 2, 6n - 2\}$, respectively. Thus, $z \notin N_{H_1}(x)$ and $y = N_{H_1}(x) \cap N_{H_1}(z)$. Hence, $|C| \geq 5$. All the other cases are analogous. The circuit $(1, 2, 3, 3n + 3, 3n + 1)$ is a 5-circuit in H_1 .

■

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows:

Set	Edges	Description
A_2	$\{(i, i + 3) i = 1, \dots, 3n - 3\}$ $\cup \{(3n - 2, 1), (3n - 1, 2), (3n, 3)\}$	Three n -circuit with weights 3 and $3n - 3$
B_2	$\{(i, i + 4) i = 3n + 1, \dots, 6n - 4\}$ $\cup \{(6n - 3, 3n + 1), (6n - 2, 3n + 2),$ $(6n - 1, 3n + 3), (6n, 3n + 4)\}$	One, two or four circuits according to the congruency of $3n$ modulo 4, with weights 4 and $3n - 4$
C_2	$\{(i, 3n + 4 + i) i = 1, \dots, 3n - 4\}$ $\cup \{(3n - 3, 3n + 1), (3n - 2, 3n + 2),$ $(3n - 1, 3n + 3), (3n, 3n + 4)\}$	Prismatic edges between W_1 and W_2 of weights 4 and $3n + 4 \equiv 3n - 3 \pmod{q}$

The graph H_2 is cubic and has weights $\pm\{3, 4, 3n - 4, 3n - 3\}$.

Lemma 5.2 *The graph H_2 has girth at least 5.*

Proof Similar to the proof of Lemma 5.1. ■

Lemma 5.3 *Let G be a graph of girth at least 5. Let $x_1x_2, x_3x_4 \in E(G)$ be two independent edges of G such that $N(x_i) \cap N(x_j) = \emptyset$, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Let $G' = G - \{x_1x_2, x_3x_4\} \cup \{(v, x_i) | i = 1, 2, 3, 4\}$ be the graph of order $|V(G)| + 1$, where $v = V(G') - V(G)$. Then G' has girth at least 5.*

Proof Let C be the shortest circuit in G' . If $E(C) \subset E(G)$ then, by hypothesis, $|C| > 4$. Otherwise $v \in V(C)$ and $x_i v x_j$ is a path in C for some $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. In G' the set

$\{x_i | i = 1, 2, 3, 4\}$ is independent, so $|C| > 3$. By hypothesis, $N(x_i) \cap N(x_j) = v$ in G' and hence $|C| > 4$. ■

Let G_1 be a graph on q vertices labelled from 0 through $q - 1$ and defined as follows $G_1 := H_1 - \{(1, 3n), (\lfloor \frac{3n+1}{2} \rfloor, 3n + \lfloor \frac{3n+1}{2} \rfloor)\} \cup \{(0, 1), (0, \lfloor \frac{3n+1}{2} \rfloor), (0, 3n), (0, 3n + \lfloor \frac{3n+1}{2} \rfloor)\}$

Lemma 5.4 *The graph G_1 has girth at least 5.*

Proof The edges $e_1 = (1, 3n)$ and $e_2 = (\lfloor \frac{3n+1}{2} \rfloor, 3n + \lfloor \frac{3n+1}{2} \rfloor)$ are independent in H_1 . The neighbourhoods of the endvertices of e_1 and e_2 are:

$$\begin{aligned} N(1) &= \{2, 3n, 3n + 1\}; \\ N(\lfloor \frac{3n+1}{2} \rfloor) &= \{\lfloor \frac{3n+1}{2} \rfloor - 1, \lfloor \frac{3n+1}{2} \rfloor + 1, 3n + \lfloor \frac{3n+1}{2} \rfloor\}; \\ N(3n) &= \{1, 3n - 1, 6n\}; \\ N(3n + \lfloor \frac{3n+1}{2} \rfloor) &= \{3n + \lfloor \frac{3n+1}{2} \rfloor - 1, 3n + \lfloor \frac{3n+1}{2} \rfloor + 1, \lfloor \frac{3n+1}{2} \rfloor\}; \end{aligned}$$

which satisfy the hypothesis of Lemma 5.3. Since G_1 is constructed from H_1 as G' from G in Lemma 5.3, we can conclude that G_1 has girth at least 5. ■

All together the weights of H_1 and G_1 modulo p give

$$\mathcal{P}_\omega := \begin{cases} \pm\{1, 2, \frac{3n+1}{2}, 3n - 2, 3n - 1, 3n\} & \text{if } n \text{ is odd} \\ \pm\{1, 2, \frac{3n}{2}, \frac{3n+2}{2}, 3n - 2, 3n - 1, 3n\} & \text{if } n \text{ is even} \end{cases} \quad (1)$$

Let G_2 be a graph on q vertices labelled from 0 through $q - 1$ and defined as follows:

$$G_2 := \begin{cases} H_2 - \{(3, 22), (5, 24)\} \cup \{(0, 3), (0, 5), (0, 22), (0, 24)\} & \text{if } n = 5 \\ H_2 - \{(3, 3n + 7), (4, 3n + 8)\} & \text{if } n \geq 6 \\ \cup \{(0, 3), (0, 4), (0, 3n + 7), (0, 3n + 8)\} & \end{cases}$$

Note that for $n = 5$ the edge $(0, 3n + 8) = (0, 23)$ has weight -8 which lies already in \mathcal{P}_ω and Theorem 4.1 cannot be applied. This is why, in the definition of G_2 , we choose to delete the edge $(5, 24)$ from H_2 , instead of $(4, 3n + 8) = (4, 23)$.

Lemma 5.5 *The graph G_2 has girth at least 5.*

Proof First suppose $n \geq 6$. As in Lemma 5.4, the edges $(3, 3n + 7), (4, 3n + 8)$ are independent in H_2 and the neighbourhoods

$$\begin{aligned} N(3) &= \{6, 3n, 3n + 7\}; \\ N(\lfloor \frac{3n+1}{2} \rfloor) &= \{\lfloor \frac{3n+1}{2} \rfloor - 1, \lfloor \frac{3n+1}{2} \rfloor + 1, 3n + \lfloor \frac{3n+1}{2} \rfloor\}; \\ N(3n) &= \{1, 3n - 1, 6n\}; \\ N(3n + \lfloor \frac{3n+1}{2} \rfloor) &= \{3n + \lfloor \frac{3n+1}{2} \rfloor - 1, 3n + \lfloor \frac{3n+1}{2} \rfloor + 1, \lfloor \frac{3n+1}{2} \rfloor\}; \end{aligned}$$

which satisfy the hypothesis of Lemma 5.3. Since G_2 is constructed from H_2 as G' from G in Lemma 5.3, G_2 has girth at least 5.

Similarly for $n = 5$. ■

All together the weights of H_2 and G_2 modulo q give

$$\mathcal{L}_\omega := \begin{cases} \pm\{3, 4, 7, 9, 11, 12\} & \text{if } n = 5 \\ \pm\{3, 4, 3n - 7, 3n - 6, 3n - 4, 3n - 3\} & \text{if } n \geq 6 \end{cases} \quad (2)$$

Theorem 5.6 *Let q be a prime such that $q = 6n + 1$, $n \geq 2$. Then, there is a $(q + 3 - u)$ -regular graph of girth 5 and order $2(q^2 - u - 1)$, for each $0 \leq u \leq q - 1$.*

Proof We treat the cases $n = 2, 3$ in Section 6. For $n = 4$, $q = 6n + 1 = 25$ is not a prime, therefore we can assume that $n \geq 5$.

Let $S = T = \{0\}$ and choose H_i, G_i for $i = 1, 2$ as previously described in this subsection. Lemmas 5.1, 5.2, 5.4, 5.5 together with (1) and (2) imply that the hypothesis of Theorem 4.1 are satisfied. Therefore, the graphs $B_q^*(S, T, u)$ are $(q + 3 - u)$ -regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q - 1$. Note that the girth of $B_q^*(S, T, u)$ is exactly 5 because H_1 has girth 5 (cf. Remark 4.2). ■

5.2 Construction for primes $q = 6n + 5$

We consider $n \geq 3$ throughout this subsection and we treat smaller cases in Section 6. Let H_1 and H_2 be two graphs of order $q - 1$ with the vertices labelled from 1 through $6n + 4$, and partitioned into $W_1 = \{1, 2, \dots, 3n + 2\}$ and $W_2 = \{3n + 3, \dots, 6n + 4\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows:

Set	Edges	Description
A_1	$\{(i, i + 1) i = 1, \dots, 3n + 1\} \cup \{(3n + 2, 1)\}$	$(3n + 2)$ -circuit with weights 1 and $3n + 1$
B_1	$\{(i, i + 2) i = 3n + 3, \dots, 6n + 2\} \cup \{(6n + 3, 3n + 3), (6n + 4, 3n + 4)\}$	one or two circuits according to the parity of n , with weights 2 and $3n$
C_1	$\{(i, 3n + i + 2) i = 1, \dots, 3n + 2\}$	Prismatic edges between W_1 and W_2 of weight $3n + 2$

The graph H_1 is cubic and has weights $\pm\{1, 2, 3n, 3n + 1, 3n + 2\}$.

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows:

Set	Edges	Description
A_2	$\{(i, i+3) i = 1, \dots, 3n-1\}$ $\cup \{(3n, 1), (3n+1, 2), (3n+2, 3)\}$	One $3n+2$ -circuit with weights 3 and $3n-1$
B_2	$\{(i, i+4) i = 3n+3, \dots, 6n\}$ $\cup \{(6n+1, 3n+3), (6n+2, 3n+4),$ $(6n+3, 3n+5), (6n+4, 3n+6)\}$	One, two or four circuits according to the congruency of n modulo 4, with weights 4 and $3n-2$
C_2	$\{(i, 3n+i+6) i = 1, \dots, 3n-2\}$ $\cup \{(3n-1, 3n+3), (3n, 3n+4),$ $(3n+1, 3n+5), (3n+2, 3n+6)\}$	Prismatic edges between W_1 and W_2 of weights 4 and $3n+6 \equiv 3n-1 \pmod{q}$

The graph H_2 is cubic and has weights $\pm\{3, 4, 3n-2, 3n-1\}$.

Let G_1 be a graph on q vertices labeled from 0 through $q-1$ and defined as follows:

$$G_1 := \begin{cases} H_1 - \{(1, 12), (6, 17)\} \cup \{(0, 1), (0, 6), (0, 12), (0, 17)\} & \text{if } n = 3 \\ H_1 - \{(1, 3n+3), (\lfloor \frac{3n+1}{2} \rfloor, 3n+2 + \lfloor \frac{3n+1}{2} \rfloor)\} \\ \cup \{(0, 1), (0, \lfloor \frac{3n+1}{2} \rfloor), (0, 3n+3), (0, 3n+2 + \lfloor \frac{3n+1}{2} \rfloor)\} & \text{if } n \geq 4 \end{cases}$$

Note that for $n = 3$ the independent edges $(1, 3n+3) = (1, 12)$ and $(\lfloor \frac{3n+1}{2} \rfloor, 3n+2 + \lfloor \frac{3n+1}{2} \rfloor) = (5, 6)$ of H_1 have a common neighbour, namely $N_{H_1}(12) \cap N_{H_1}(16) = \{14\}$, and Lemma 5.3 cannot be applied. This is why we choose the independent edges $(1, 12)$ and $(6, 17)$ in H_1 with pairwise disjoint neighbourhoods to define G_1 .

All together the weights of H_1 and G_1 modulo q give

$$\mathcal{P}_\omega := \begin{cases} \pm\{1, 2, 6, 9, 10, 11\} & \text{if } n = 3 \\ \pm\{1, 2, \frac{3n+1}{2}, \frac{3n+5}{2}, 3n, 3n+1, 3n+2\} & \text{if } n \text{ is odd and } n \geq 5 \\ \pm\{1, 2, \frac{3n}{2}, \frac{3n+6}{2}, 3n, 3n+1, 3n+2\} & \text{if } n \text{ is even} \end{cases} \quad (3)$$

Let G_2 be a graph on q vertices labeled from 0 through $q-1$ and defined as follows $G_2 := H_2 - \{(3, 3n+9), (4, 3n+10)\} \cup \{(0, 3), (0, 4), (0, 3n+9), (0, 3n+10)\}$.

All together the weights of H_2 and G_2 modulo q give

$$\mathcal{L}_\omega := \pm\{3, 4, 3n-5, 3n-4, 3n-2, 3n-1\}. \quad (4)$$

Lemma 5.7 *The graphs H_1, H_2, G_1 and G_2 have girth at least 5.*

Proof Similar to Lemmas 5.1, 5.2, 5.4, 5.5. ■

Note that in general, the girth of H_1 is exactly 5, since $(1, 2, 3, 3n+5, 3n+3)$ is a 5-circuit in H_1 .

Theorem 5.8 *Let q be a prime such that $q = 6n+5$, for $n \geq 3$. Then, there is a $(q+3-u)$ -regular graph of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q-1$.*

Proof Let $S = T = \{0\}$ and choose H_i, G_i for $i = 1, 2$ as previously described in this subsection. By (3), (4) and Lemma 5.7, all the hypothesis of Theorem 4.1 are satisfied. Thus, the graphs $B_q^*(S, T, u)$ are $(q + 3 - u)$ -regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q - 1$. Note that the girth of $B_q^*(S, T, u)$ is exactly 5 because H_1 has girth 5 (cf. Remark 4.2). ■

6 Small Cases

We now present some constructions of graphs $B_q^*(S, T, u)$ for small prime values of q . The first two constructions complete the proof of Theorem 5.6.

6.1 $q = 13$

In this case, let $S = T = \{0\}$, H_1, H_2, G_1 and G_2 as in Figure 1. The graphs G_i are obtained from H_i deleting two independent edges satisfying the hypothesis of Lemma 5.3 and joining all their end-vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (3, 12)\} \cup \{(0, 1), (0, 3), (0, 10), (0, 12)\}$, $G_2 = H_2 - \{(2, 8), (5, 11)\} \cup \{(0, 2), (0, 8), (0, 5), (0, 11)\}$ and as unlabeled graphs G_1 is isomorphic to G_2 . Hence, the graphs G_1 and G_2 have order 13, girth 5 and are bi-regular with one vertex of degree four and all other vertices of degree three.

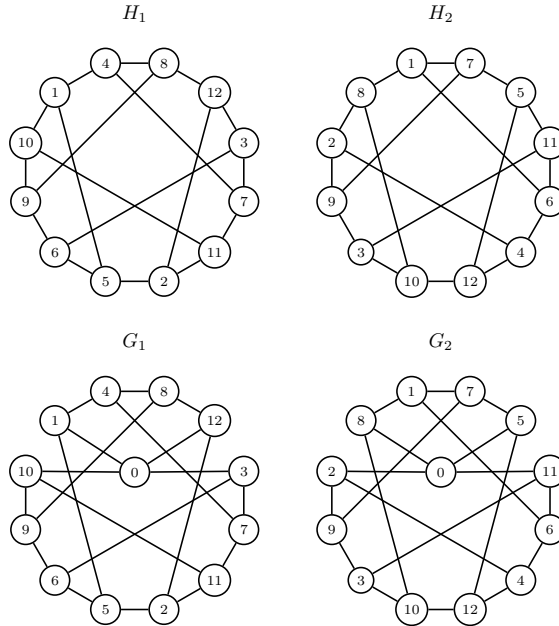


Figure 1: The graphs H_i and G_i for $i \in 1, 2$ and $q = 13$.

Note that as unlabeled graphs H_1 is isomorphic to H_2 and they are both isomorphic to one of the two cubic graphs on 12 vertices of girth 5, specifically 12 *cubic graph* 84 from [26, 31].

Lemma 6.1 *Let $S = T = \{0\}$, H_1, H_2, G_1 and G_2 as described above. Then the graph $B_{13}^*(0, 0, u)$ is a $(16 - u)$ -regular graph of girth 5 and order $336 - 26u$, for $0 \leq u \leq q - 1$.*

Proof The weights of these graphs are $\mathcal{P}_\omega = \pm\{1, 3, 4\}$ and $\mathcal{L}_\omega = \pm\{2, 5, 6\}$. Thus, by Theorem 4.1, the graph $B_{13}^*(0, 0, u)$ is a $(16 - u)$ -regular graph of girth 5 and order $26(13 - u) - 2 = 336 - 26u$, for $0 \leq u \leq q - 1$. ■

- For $u = 0$, we obtain a 16-regular graph of girth 5 and order 336, with exactly the same order as the $(16, 5)$ -graphs that appear in [24, 20];
- for $u = 1$, we obtain a 15-regular graph of girth 5 and 310 vertices which has two vertices less than the $(15, 5)$ -graphs that appear in [24, 20];
- for $u = 2$ we obtain a 14-regular graph of girth 5 and 284 vertices which has four vertices less than the $(14, 5)$ -graph in [24];

6.2 $q = 19$

Let $S = T = \{0\}$ and let H_1, H_2, G_1 and G_2 be as in Figure 2. The graphs G_i are obtained from H_i deleting two independent edges satisfying the hypothesis of Lemma 5.3 and joining all their end-vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (9, 16)\} \cup \{(0, 1), (0, 9), (0, 10), (0, 16)\}$ and $G_2 = H_2 - \{(8, 13), (11, 15)\} \cup \{(0, 8), (0, 13), (0, 11), (0, 15)\}$. Hence, the graphs G_1 and G_2 have order 19, girth 5 and are bi-regular with one vertex of degree four and all other vertices of degree 3.

Lemma 6.2 *Let $S = T = \{0\}$, H_1, H_2, G_1 and G_2 be as described above. Then the graph $B_{19}^*(0, 0, u)$ is a $(22 - u)$ -regular graph of girth 5 and order $720 - 38u$, for $0 \leq u \leq q - 1$.*

Proof The weights of these graphs are $\mathcal{P}_\omega = \pm\{1, 2, 3, 7, 9\}$ and $\mathcal{L}_\omega = \pm\{4, 5, 6, 8\}$. Thus, by Theorem 4.1, the graph $B_{19}^*(0, 0, u)$ is a $(22 - u)$ -regular graph of girth 5 and order $38(19 - u) - 2 = 720 - 38u$, for $0 \leq u \leq q - 1$. ■

- For $u = 0$, we obtain a 22-regular graph of girth 5 and order 720, with exactly the same order as the $(22, 5)$ -graphs that appear in [24, 20];
- for $u = 1$, we obtain a 21-regular graph of girth 5 and 682 vertices which has two vertices less than the $(21, 5)$ -graphs that appear in [24, 20];

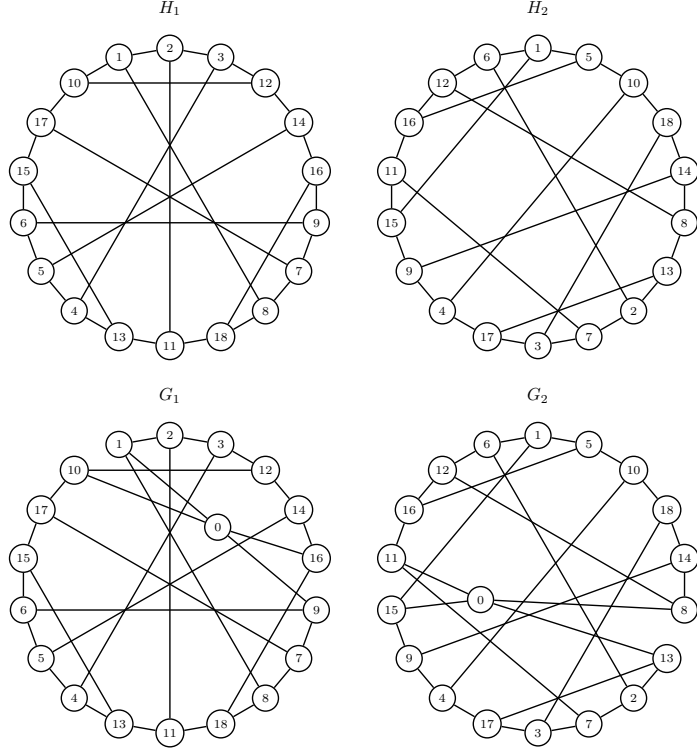


Figure 2: The graphs H_i and G_i for $i \in 1, 2$ and $q = 19$.

6.3 $q = 11$

For $q = 11$ we are going to remove 6 vertices from B_{11} instead of 2, but we will construct a $(q + 2)$ -regular graph instead of a $(q + 3)$ -regular one.

Lemma 6.3 *Let $S = \{0, 1, 2, 4, 6, 8\}$ and $T = \emptyset$. Let $H_1 = (3, 5, 10, 7, 9)$ be a 5-circuit with weights $\pm\{2, 3, 5\}$, $G_1 = (0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$ be a 11-circuit with weight $\{\pm 2\}$, and $H_2 = G_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \cup (0, 4) \cup (2, 6) \cup (1, 8)$ be a 11-circuit with three chords and weights $\pm\{1, 4\}$ (see Figure 3). Then the graph $B_{11}^*(S, T, u)$ is a $(13 - u)$ -regular graph of girth 5 and order $22(11 - u) - 6 = 236 - 22u$, for $u \leq q - 1$. In particular, we obtain a 13-regular graph of girth 5 and order 236 for $u = 0$.*

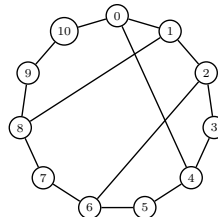


Figure 3: The graphs $H_2 = G_2$ for $q = 11$.

Proof Since $\mathcal{P}_\omega = \pm\{2, 3, 5\}$ and $\mathcal{L}_\omega = \pm\{1, 4\}$, the thesis follows by Theorem 4.1. ■

Note that the graph $B_{11}^*(S, T, 0)$ has four vertices less than those constructed in [24, 20].

6.4 $q = 17$

For $q = 17$ we are going to remove 6 vertices instead of 2 and construct a $(q + 3)$ -regular graph, obtaining a better result than the one obtained in [11].

Lemma 6.4 *Let $S = T = \{7, 10, 12\}$, H_1, H_2, G_1 and G_2 as in Figure 4. The graphs G_1 and G_2 have order 17, girth 5 and are bi-regular with three vertices of degree four and all other vertices of degree 3. Then the graph $B_{17}^*(S, T, u)$ is a $(20 - u)$ -regular graph of girth 5 and order $572 - 34u$, for $u \geq q - 1$.*

Proof In this case $\mathcal{P}_w = \pm\{1, 3, 4, 5\}$ and $\mathcal{L}_w = \pm\{2, 6, 7, 8\}$, thus, by Theorem 4.1, the graph $B_{17}^*(S, T, u)$ is a $(20 - u)$ -regular graph of girth 5 and order $34(17 - u) - 6 = 572 - 34u$ for $u \geq q - 1$. ■

In [24] the author constructs $(k, 5)$ -graphs of order $32(k - 2)$, while we have constructed $(k, 5)$ -graphs of order $34(k - 3)$ which have $44 - 2k$ fewer vertices, for $k \in \{4, \dots, 20\}$. In particular, we obtain a 20-regular graph of girth 5 and order 572 for $u = 0$ which has four vertices less than the one constructed in [24]. Note also that as unlabeled graphs $H_1 \cong H_2$ and they are both isomorphic to the Heawood graph.

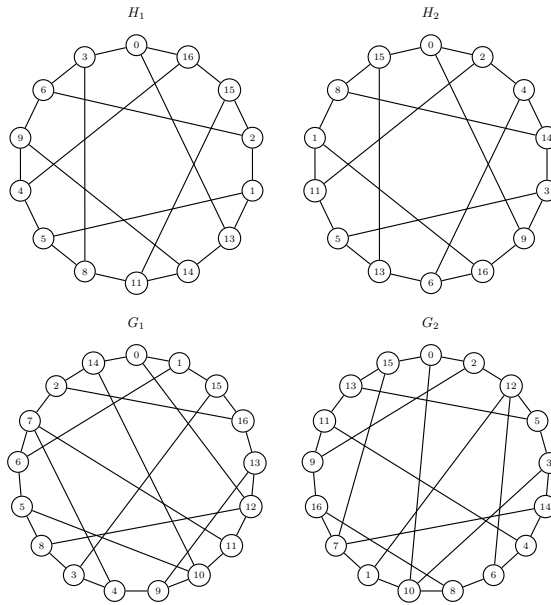


Figure 4: The graphs H_i and G_i for $i \in 1, 2$ and $q = 17$.

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